The joint gluing inequality in the language of Green's functions

An exposition of a proof by OpenAI

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Abstract

We re-present the proof of the joint gluing inequality for independent percolation from OpenAI's preprint [1, Section 6], in the vocabulary of Markov-chain potential theory. The estimates and key identities are OpenAI's. We add only a change of language and a few elementary rearrangements, flagged as such, and claim no new theorem. Two Green functions organize the argument. In the *statement*, the triangular matrix of conditioned hitting probabilities is the Green (visit) matrix of a killed Markov chain on the list of relay vertices, and the gluing inequality says that a certain function of the starting relay is excessive for this chain. In the *proof*, the positivity of that chain follows from a Poisson equation for the two-cluster Gibbs sampler of van den Berg, Häggström and Kahn [3], solved by the sampler's own Green function.

1 The inequality

Let G be a graph with countably many vertices and edges, with independent edge-open probabilities in $[0,1]$. Write $C(x)$ for the open cluster of x and $C(S) = \bigcup_{x\in S} C(x)$; connections allow paths of length zero.

Theorem 1 (Joint gluing [1, Section 6, Theorem "Joint gluing"])**.** *For a vertex o, a nonempty vertex set A, and any vertex set T,*

$$\mathbb{P}(o \leftrightarrow A,\ o \leftrightarrow T) \ \geq\ \mathbb{P}(o \leftrightarrow A)\ \inf_{a\in A} \mathbb{P}(a \leftrightarrow T).$$

With $T = \{b\}$, and after enlarging the event on the left, this is Kozma and Nitzan's multiplicative gluing inequality [2, Conjecture 1]. They reduced $\theta(p_c) = 0$ on $\mathbb{Z}^d$ to conjectured inequalities of this kind [2].

OpenAI note that the finite statement also follows from Leder's first-contact comparison (GEN) [4, Section 3.4], as recorded in [5]. Gouéré records that the Kozma–Nitzan inequality had been proved shortly before by Anthropic, by a proof relying on the same ideas [8]. Blanc-Renaudie gives a human exposition of that proof [7], and Leder's guide describes its Lean formalization [4].

We follow OpenAI's proof. Its reduction to finite graphs with all $p_e < 1$ is routine [1, Section 6]. If $o \in A$ the inequality is immediate, so assume $o \notin A$.

2 The statement: a relay chain and its Green matrix

Fix distinct vertices $x_1, \dots, x_N$, the *relays*. For $1 \leq k \leq N$ put

$$S_k = \{x_1, \dots, x_{k-1}\}, \qquad E_k = \{C(x_k) \cap S_k = \varnothing\}, \qquad d_k = \mathbb{P}(E_k) > 0.$$

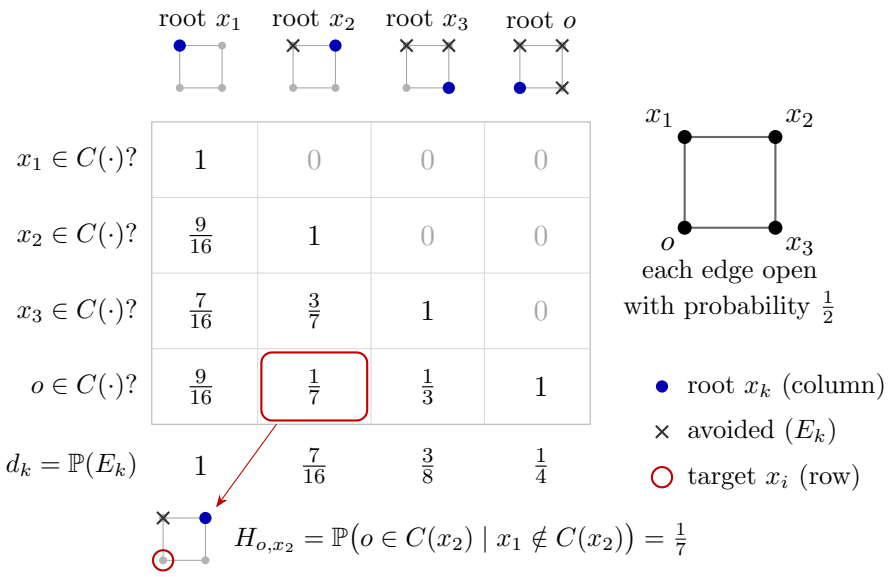


Figure 1: The matrix H for the 4-cycle with relays listed x_1, x_2, x_3, o and $p = \frac{1}{2}$, computed exactly. Column k is the cluster of x_k (blue), conditioned to avoid the earlier relays (crossed). Row i asks whether that cluster contains x_i (red ring). Entries above the diagonal vanish, since the cluster avoids x_i, and the diagonal is 1. Each row satisfies $\sum_k H_{ik} d_k = 1$. For row o the four terms $\frac{9}{16}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}$ are the law of the first listed vertex of $C(o)$.

Let μ_k be the law of $C(x_k)$ given E_k, a law on $\mathcal{D}_k = \{U \ni x_k : U \cap S_k = \varnothing\}$. For $F : \mathcal{D}_k \to \mathbb{R}$ let

$$v_k^F(i) = \mathbb{E}_{\mu_k}\big[\mathbf{1}_{\{x_i \in C(x_k)\}} F(C(x_k))\big], \qquad h_k = v_k^1, \qquad H = (h_1 \ \cdots \ h_N).$$

Thus $H_{ik} = \mathbb{P}(x_i \in C(x_k) \mid E_k)$, and H is unit lower triangular (Figure 1). Write e_k for the kth coordinate vector.

The organizing fact is the *first-label partition*. For $k \le i$, the event that x_k is the first listed vertex of $C(x_i)$ is $E_k \cap \{x_i \in C(x_k)\}$, and on it $C(x_i) = C(x_k)$. Hence, for any function F of vertex sets [1, Section 6.3],

$$\mathbb{P}\big(\text{first listed vertex of } C(x_i) \text{ is } x_k\big) = H_{ik}\, d_k, \qquad \mathbb{E}\, F(C(x_i)) = \sum_k d_k\, v_k^F(i), \qquad Hd = \mathbf{1}. \quad (1)$$

Define the strictly lower-triangular matrix

$$\Gamma = I - H^{-1}, \qquad \text{so that} \qquad H = \sum_{n \ge 0} \Gamma^n \quad \text{(a finite sum)}, \qquad (I - \Gamma)\mathbf{1} = d.$$

Proposition 2 (Relay positivity [1, Section 6.3])**.** $\Gamma \ge 0$ *entrywise. The proof is recalled in Section 3.*

The relay chain (a rereading). Granting Proposition 2, row i of Γ consists of nonnegative numbers Γ_{ij}, $j < i$, with $\sum_j \Gamma_{ij} = 1 - d_i$. So Γ is the transition kernel of a Markov chain on $\{1, \dots, N\}$. From relay i it jumps to an earlier relay j with probability Γ_{ij}, and is killed with probability d_i. Indices strictly decrease, so each state is visited at most once, and

$$H_{ik} = \sum_{n \ge 0} (\Gamma^n)_{ik} = \mathbb{P}_i(\text{the chain visits } k).$$

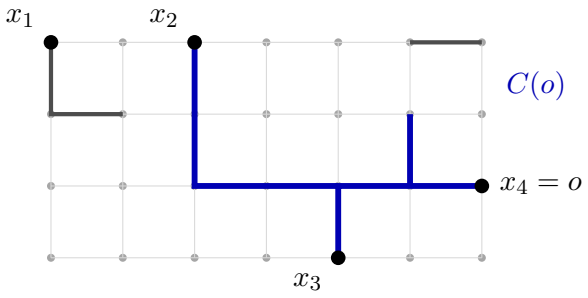


Figure 2: The first-label partition. The relays are listed x_1, x_2, x_3 (the set A), then $x_4 = o$. Thick edges are open, and the cluster $C(o)$ is drawn in blue. It contains x_2 and x_3 but not x_1, so its first listed vertex is x_2. This is the event $E_2 \cap \{o \in C(x_2)\}$, and summing over such events gives $\mathbb{P}(\text{first listed vertex of } C(x_i) \text{ is } x_k) = H_{ik} d_k$.

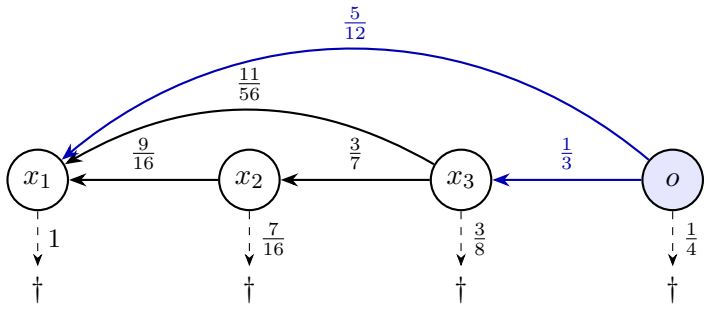


Figure 3: The relay chain $\Gamma = I - H^{-1}$ of the example in Figure 1. Solid arrows carry the jump probabilities Γ_{ij}; there is no arrow $o \to x_2$, because $\Gamma_{o,x_2} = 0$. Dashed arrows carry the killing probabilities d_i. The Green-matrix reading checks out: $H_{o,x_2} = \mathbb{P}_o(\text{visit } x_2) = \frac{1}{3} \cdot \frac{3}{7} = \frac{1}{7}$, and $\sum_j \Gamma_{oj} = \frac{5}{12} + \frac{1}{3} = \frac{3}{4} = \mathbb{P}(o \leftrightarrow A)$. Joint gluing uses one step from o (blue arrows): $y_o \geq \sum_j \Gamma_{oj}\, y_j$.

In words: *H is the Green function of the relay chain.* Its killing-site law $\mathbb{P}_i(\text{killed at } k) = H_{ik} d_k$ coincides, by (1), with the law of the first listed vertex of $C(x_i)$. This identity is used only at the level of laws; no pathwise coupling of the chain with the clusters is claimed or used.

The inequality as excessiveness. The full positivity statement is Lemma 4 below: $H^{-1} v_k^F \geq 0$ for every increasing $F \geq 0$ on $\mathcal{D}_k$. By (1), the vector $y_i = \mathbb{E}F(C(x_i))$ then satisfies

$$(I - \Gamma)\, y = H^{-1} y = \sum_k d_k\, H^{-1} v_k^F \;\geq\; 0, \qquad \text{that is,} \qquad y \;\geq\; \Gamma y.$$

Gouéré states this row inequality in exactly this form [8, Theorem 2]. In potential-theoretic terms, y is *excessive* (superharmonic) for the relay chain. Equivalently, $y = Hm$ is the Green potential of the nonnegative charge $m = (I - \Gamma)y$.

Theorem 1 is one step of this. List the vertices of A first and $x_N = o$ last, and take $F(U) = \mathbf{1}_{\{U \cap A \neq \varnothing,\, U \cap T \neq \varnothing\}}$. Then:

- $y_i = \mathbb{P}(x_i \leftrightarrow T)$ for $x_i \in A$;
- y_N is the left side of Theorem 1;

- $\sum_{i<N} \Gamma_{Ni} = 1 - d_N = \mathbb{P}(o \leftrightarrow A)$.

Hence

$$y_N \;\geq\; (\Gamma y)_N = \sum_{i<N} \Gamma_{Ni}\, y_i \;\geq\; \mathbb{P}(o \leftrightarrow A) \min_{a \in A} \mathbb{P}(a \leftrightarrow T).$$

The relay weights of [1] are the first-step probabilities of the relay chain from o.

Remark 3. Kozma and Nitzan sketched a related system of coefficients for a different inequality (their Question 5) [2, Section 5.2]. It was built from a symmetric matrix of joint connection probabilities, and they report that the approach "did not pan out" and that some coefficients can be negative once $|A| \geq 4$. OpenAI point to this contrast [1, Section 6]. With their ordered, triangular matrix, the coefficients are the transition probabilities of a Markov chain precisely when Proposition 2 holds. Proving that positivity is the content of the argument.

3 The proof: a Poisson equation for the Gibbs sampler

Lemma 4 (Cluster-column positivity [1, Section 6.2])**.** *For every k and every increasing $F \geq 0$ on $\mathcal{D}_k$,*

$$\kappa_k^F := v_k^F - \mu_k(F)\, h_k \in \operatorname{cone}\{h_l : l > k\},$$

and hence $H^{-1} v_k^F \geq 0$. Since κ_k^F is unchanged when a constant is added to F, the first statement also holds for every real-valued increasing F. This extension is elementary; OpenAI state the lemma for $F \geq 0$.

Proposition 2 follows by taking $F(U) = \mathbf{1}_{\{U \cap \{x_{k+1}, \ldots, x_N\} \neq \varnothing\}}$. Its mean satisfies $\mu_k(F) < 1$, since x_k can be isolated. Moreover $\kappa_k^F = (1 - \mu_k(F))(h_k - e_k)$, so $h_k = e_k + \sum_{l>k} \Gamma_{lk} h_l$ with $\Gamma_{lk} \geq 0$ [1, Section 6.3]. Note that $v_l^F = \mu_l(F) h_l + \kappa_l^F$ for every l.

Three ingredients. Fix $k < N$ and write $s = x_k$, $S = S_k$. For $w \supseteq S$ with $s \notin w$ let $c(w) = \mathbb{E}^{G \setminus w} F(C(s))$. For U disjoint from S let $W_U = C^{G \setminus U}(S)$. All three ingredients are in [1, Section 6.2].

1. *The two-cluster Gibbs sampler* of van den Berg, Häggström and Kahn [3, Section 2.1].

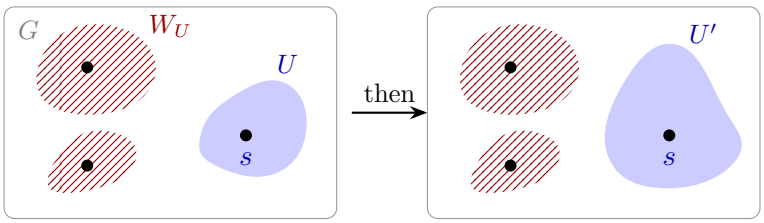
   - On $\mathcal{D}_k$, the transition P samples W_U and then a fresh cluster of s in $G \setminus W_U$; so $(PF)(U) = \mathbb{E}\, c(W_U)$.
   - The law μ_k is stationary, since both half-steps are conditional laws under $\mathbb{P}(\cdot \mid E_k)$.
   - P maps increasing functions to increasing functions.
   - Every transition gives mass at least $\delta = \prod_{e \ni s}(1 - p_e) > 0$ to $\{s\}$, so $\operatorname{osc}(P^j g) \leq (1 - \delta)^j \operatorname{osc}(g)$.

2. *The deleted-set residual.* For $U \subseteq V \setminus (S \cup \{s\})$ put $R_F(U) = \mathbb{E}\big[\mathbf{1}_{\{s \notin W_U\}} \big(c(W_U) - c(W_U \cup U)\big)\big]$. If F is increasing, then $R_F \geq 0$ and R_F is increasing.

3. *The column identity.*

$$v_k^F = v_k^{PF} + \sum_{l>k} \frac{d_l}{d_k}\, v_l^{R_F}. \tag{2}$$

   OpenAI derive it for $F \geq 0$. The derivation uses neither the sign nor the monotonicity of F, and both sides are linear in F, so (2) holds for every real F.



(a) sample $W_U = C^{G\setminus U}(S)$ (b) fresh $C(s)$ in $G \setminus W_U$

Figure 4: One step $U \mapsto U'$ of the two-cluster Gibbs sampler of van den Berg, Häggström and Kahn [3], with $s = x_k$ and $S = \{x_1, \dots, x_{k-1}\}$ (black dots on the left). (a) Given the cluster U of s (blue), resample the clusters W_U of the earlier relays in $G \setminus U$ (hatched). (b) Given W_U, resample the cluster of s in $G \setminus W_U$. Thus $(PF)(U) = \mathbb{E}\, c(W_U)$ with $c(w) = \mathbb{E}^{G\setminus w} F(C(s))$. The law μ_k is stationary, and step (b) isolates s with probability at least δ.

The Poisson equation (a rearrangement). By the Doeblin bound in item 1, the series

$$\Psi = \sum_{j\ge 0} \big(P^j F - \mu_k(F)\big)$$

converges uniformly, and it is increasing when F is, because each $P^j F$ is. It solves the Poisson equation

$$\Psi - P\Psi = F - \mu_k(F).$$

In Markov-chain terms, $\Psi = \mathcal{G}(F - \mu_k(F))$ with $\mathcal{G} = \sum_{j\ge 0}(P^j - \Pi) = Z - \Pi$. Here Π is the projection onto constants under μ_k, and $Z = (I - P + \Pi)^{-1}$ is Kemeny and Snell's fundamental matrix [9]. The two agree on centred functions.

Proof of Lemma 4. The proof is by downward induction on k; for $k = N$ the vector κ_N^F vanishes. Let $k < N$, assume the lemma for all larger indices, and let F be increasing. Identity (2) is linear in F, and $v_k^{\mathrm{const}} = \mathrm{const} \cdot h_k$. Applying it to Ψ gives

$$\kappa_k^F = v_k^{F-\mu_k(F)} = v_k^{\Psi - P\Psi} = \sum_{l>k} \frac{d_l}{d_k}\, v_l^{R_\Psi}.$$

For $l > k$, every set in $\mathcal{D}_l$ avoids $S_l \supseteq S \cup \{s\}$, so R_Ψ is defined there. By item 2 it is nonnegative and increasing. By the induction hypothesis, $v_l^{R_\Psi} = \mu_l(R_\Psi) h_l + \kappa_l^{R_\Psi} \in \mathrm{cone}\{h_j : j \ge l\}$. Hence $\kappa_k^F \in \mathrm{cone}\{h_l : l > k\}$. □

What the Green-function form does, and does not, change. OpenAI iterate (2): since $v_k^F - v_k^{P^j F}$ lies in the closed cone for every j, letting $j \to \infty$ proves the lemma [1, Section 6.2]. The Poisson form sums that iteration once, into Ψ. The Doeblin bound is still what makes the sum converge; only the closed-cone limit disappears. The proof is no shorter.

The alternating sampler and its iteration go back to van den Berg, Häggström and Kahn [3].

- Leder uses the same scheme in step (T) of his guide to the separate, formalized proof, crediting [3, proof of Theorem 2.1] [4].
- Analogous iterations, though not the same sums, appear in two expositions of that proof: the lecture notes [6, Theorem 4.9, Step 1] and Blanc-Renaudie [7, Proposition 10].

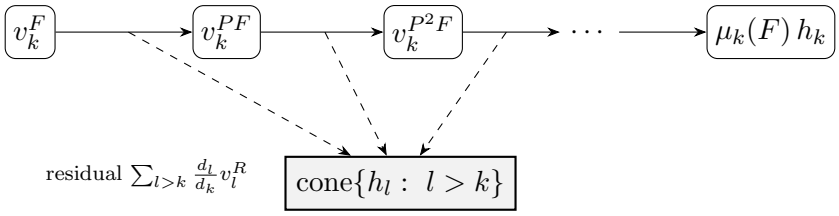


Figure 5: The proof technique. Moving along the top row applies the Gibbs sampler, and the Doeblin bound makes the row converge to $\mu_k(F)h_k$. OpenAI: each step of (2) sends a nonnegative residual into the closed cone of later columns (by the deleted-set residual and the induction hypothesis), so $v_k^F - \mu_k(F)h_k$ lies in the cone in the limit. Poisson form: $\Psi = \sum_j (P^j F - \mu_k F)$ solves $\Psi - P\Psi = F - \mu_k F$, so a single application of (2) gives $\kappa_k^F = \sum_{l>k} \frac{d_l}{d_k} v_l^{R_\Psi}$.

- One also appears in Gouéré's exposition of OpenAI's cubic-lattice preprint [8].

What the Green-function form offers is vocabulary that probabilists who work with random walks already have:

- the statement says that the hitting matrix is the Green matrix of a killed chain, and the inequality is excessiveness;
- the proof says that the obstruction, the dependence between a relay's cluster and the earlier clusters, is removed by solving one Poisson equation for the Gibbs sampler that resamples them.

Remark 5 (Explicit coefficients, a rearrangement). Write $\alpha^{(k)}(F) = H^{-1}\kappa_k^F$ for the cone coefficients of column k. Let $\Psi = \Psi_k(F)$ and let $R = R^{(k)}$ be the residual with the roots x_k, S_k of level k; inside $\alpha^{(l)}(\cdot)$ the residuals are those of level l. Unwinding the proof gives, for $j > k$,

$$\alpha_j^{(k)}(F) = \frac{d_j}{d_k}\,\mu_j\big(R_\Psi^{(k)}\big) + \sum_{k<l<j} \frac{d_l}{d_k}\,\alpha_j^{(l)}\big(R_\Psi^{(k)}\big).$$

In words, column k's coefficients are the later columns' coefficients of the residual of the sampler's potential.

Credit. The estimates and identities of Sections 2 and 3 are OpenAI's [1, Section 6]:

- the first-label partition, $H^{-1} = I - \Gamma$, $(I - \Gamma)\mathbf{1} = d$ and $\Gamma \geq 0$;
- the column identity and the deleted-set residual;
- the Doeblin bound.

The sampler and its iteration are due to van den Berg, Häggström and Kahn [3]. Gouéré states the row inequality $y \geq \Gamma y$ [8, Theorem 2]. The Poisson equation and the fundamental matrix are classical [9]. Our additions are elementary rearrangements:

- the Green-matrix reading of H, at the level of laws;
- the Poisson form of the iteration;
- the coefficient recursion;
- the extensions to real-valued F.

AI use. AI was used in the preparation of this note.

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